

The Center of the Nilcoxeter and 0-Hecke Algebras

Joelle Brichard

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Abstract

We compute the dimension of the center of the 0-Hecke algebra \mathcal{H}_n and of the Nilcoxeter algebra \mathcal{NC}_n using a calculus of diagrams on the Möbius band. In the case of the Nilcoxeter algebra, this calculus is shown to produce a basis for $Z(\mathcal{NC}_n)$ and the table of multiplication in this basis is shown to be trivial. We conjecture that a basis for $Z(\mathcal{H}_n)$ can also be obtained in a specific way from this topological calculus.

1 Introduction

1.1 Generic Algebras

Given a Coxeter system (W, S) and an associative commutative algebra \mathcal{A} with unity, we start by defining the generic algebra $\mathcal{E}_{\mathcal{A}}(a_s, b_s)$. This approach is taken from [5] and depends on parameters $\{a_s, b_s \in \mathcal{A}\}_{s \in S}$ subject to the condition that $a_s = a_t$ and $b_s = b_t$ whenever s and t are conjugates in W . $\mathcal{E}_{\mathcal{A}}(a_s, b_s)$ is generated by elements T_w for $w \in W$ with relations

1. $T_s T_w = T_{sw}$ if $l(sw) > l(w)$
2. $T_s T_w = a_s T_w + b_s T_{sw}$ if $l(sw) < l(w)$.

The two algebras of concern in this paper are both examples of this type of generic algebras with Coxeter group \mathcal{S}_n . Other important examples include the group algebra $\mathcal{A}[W]$ which is obtained by setting all $a_s = 0$ and $b_s = 1$. Hecke algebras are also examples of this construction for $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$ and with $a_s = q - 1$ and $b_s = q$ for all $s \in S$. For more on Coxeter groups and Hecke algebras see [6].

There is another set of relations equivalent to the ones above:

1. $T_s T_w = T_{sw}$ if $l(sw) < l(w)$,
2. $T_s^2 = a_s T_s + b_s T_e$

with T_e the identity element. \mathcal{E} is in fact generated as an algebra by the T_s , $s \in S$ and $1 = T_e$.

Any generic algebra with coxeter group \mathcal{S}_n , is generated by the following three relations:

1. $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$
2. $T_i T_j = T_j T_i$ for $|i - j| > 1$
3. $T_i^2 = a_i T_i + b_i T_e$

1.1.1 The Nilcoxeter Algebra

The Nilcoxeter algebra, which we denote \mathcal{NC}_n , is an example of a generic algebra associated to \mathcal{S}_n where we set $a_s = b_s = 0$ for all $s \in S$. This algebra first appeared in [1] in relation to the cohomology of flag varieties. It was also studied by Lascoux and Schützenberger [10], Macdonald [11], Fomin and Stanley [4] and others. Khovanov has shown in [7] that it categorifies the polynomial representation of the Weyl algebra and recent work by Khovanov and Lauda showed its relevance to the categorification of quantum groups [8]. In terms of the generators T_i , where $T_i = T_{s_i}$ for $s_i \in \mathcal{S}_n$ the transposition $(i \ i + 1)$, we have the following defining relations:

1. $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$
2. $T_i T_j = T_j T_i$ for $|i - j| > 1$
3. $T_i^2 = 0$

We therefore also have

$$T_i T_w = \begin{cases} T_{s_i w} & \text{if } l(s_i w) > l(w) \\ 0 & \text{if } l(s_i w) < l(w) \end{cases}.$$

1.1.2 The 0-Hecke Algebra

The 0-Hecke algebra, denoted \mathcal{H}_n , is the Hecke algebra associated to \mathcal{S}_n with $q = 0$. Equivalently, it is the generic algebra where we have set $a_s = 1$ and $b_s = 0$ for all $s \in S$. That is, the third relation on the T_i now becomes

$$T_i^2 = T_i.$$

This implies that

$$T_i T_w = \begin{cases} T_{s_i w} & \text{if } l(s_i w) > l(w) \\ T_w & \text{if } l(s_i w) < l(w) \end{cases}$$

Norton studied the Hecke algebra of a Coxeter group at $q = 0$ for all types in [12]; she classified the irreducible modules, described the decomposition of the algebra into left ideals as well as the Cartan invariants. In [2], Carter gave decomposition numbers for \mathcal{H}_n in the type A case. Krob and Thibon gave a representation-theoretic interpretation of non-commutative algebras [9] and works by Duchamp, Hivert and Thibon [3] showed that the representations of \mathcal{H}_n in type A categorify the ring of quasi-symmetric functions.

1.2 The Centers of Frobenius Algebras

We now derive some general facts about the centers of Frobenius algebras and then introduce a trace map which makes both \mathcal{NC}_n and \mathcal{H}_n into Frobenius algebra. The facts here derived are crucial to our approach to describing the centers of \mathcal{NC}_n and \mathcal{H}_n .

1.2.1 Symmetric Frobenius Algebras

To investigate the centers of Frobenius algebras, it is sometimes convenient to use a duality relation between the center $Z(\mathcal{A})$ and a quotient of \mathcal{A} by certain subgroups. In the case of symmetric Frobenius algebras, this subgroup is simply the commutator subgroup. Let \mathcal{A} be a Frobenius algebra over a field k , $\varepsilon : \mathcal{A} \rightarrow k$ its trace map and assume that \mathcal{A} is a symmetric algebra, meaning that $\varepsilon(ab) = \varepsilon(ba)$ for all a, b . Denote by $\mathcal{A}^* = \text{Hom}_k(\mathcal{A}, k)$ the dual of \mathcal{A} . Then ε extends to an isomorphism of \mathcal{A} -bimodules:

$$\begin{aligned} \varphi : \mathcal{A} &\rightarrow \mathcal{A}^* \\ 1 &\mapsto \varepsilon \\ a &\mapsto \varepsilon(*a). \end{aligned}$$

Let $[\mathcal{A}, \mathcal{A}]$ denote the commutator subspace of \mathcal{A} : the space spanned by elements of the form $ab - ba$ for $a, b \in \mathcal{A}$.

Claim 1.1. If \mathcal{A} is a symmetric Frobenius algebra, then

$$(\mathcal{A}/[\mathcal{A}, \mathcal{A}])^* \simeq Z(\mathcal{A}).$$

Proof. Let

$$\begin{aligned} \Psi : Z(\mathcal{A}) &\rightarrow (\mathcal{A}/[\mathcal{A}, \mathcal{A}])^* \\ z &\mapsto z\varepsilon : \mathcal{A} \rightarrow k; z\varepsilon(a) = \varepsilon(za) = \varepsilon(az). \end{aligned}$$

Ψ is well-defined:

$$z\varepsilon([a, b]) = z\varepsilon(ab - ba) = \varepsilon(zab - zba) = \varepsilon(bza) - \varepsilon(zba) = \varepsilon(zba) - \varepsilon(zba) = 0$$

since z is a central element and \mathcal{A} is symmetric.

Ψ is injective: since the map $\mathcal{A} \rightarrow \mathcal{A}^*$ is an isomorphism and the map $Z(\mathcal{A}) \rightarrow \mathcal{A}$ is an inclusion, the composition $Z(\mathcal{A}) \rightarrow \mathcal{A} \rightarrow \mathcal{A}^*$ is injective. Moreover, we have seen that the map $Z(\mathcal{A}) \rightarrow \mathcal{A}^*$ factors through $Z(\mathcal{A}) \rightarrow (\mathcal{A}/[\mathcal{A}, \mathcal{A}])^*$, which means that $\Psi : Z(\mathcal{A}) \rightarrow (\mathcal{A}/[\mathcal{A}, \mathcal{A}])^*$ is injective as well.

Ψ is surjective: let $\alpha \in (\mathcal{A}/[\mathcal{A}, \mathcal{A}])^*$. Because

$$\pi : \mathcal{A} \rightarrow \mathcal{A}/[\mathcal{A}, \mathcal{A}]$$

is surjective, its dual

$$\pi^*(\mathcal{A}/[\mathcal{A}, \mathcal{A}])^* \rightarrow \mathcal{A}^*$$

must also be injective. We know from the isomorphism $\varphi : \mathcal{A} \rightarrow \mathcal{A}^*$ that $\pi^*(\alpha) = z\varepsilon$ for some $z \in \mathcal{A}$. We must check that $z \in Z(\mathcal{A})$. Now, since $\alpha \in (\mathcal{A}/[\mathcal{A}, \mathcal{A}])^*$, $z\varepsilon([a, b]) = 0$ for all $a, b \in \mathcal{A}$. Hence

$$\varepsilon(z[a, b]) = \varepsilon(zab) - \varepsilon(zba) = \varepsilon(zab) - \varepsilon(azb) = \varepsilon([z, a]b) = 0.$$

Since the trace ε is non-degenerate, this cannot be true for all $b \in \mathcal{A}$ unless $[z, a] = 0$. This is true for all $a \in \mathcal{A}$ as well, so we conclude that $z \in Z(\mathcal{A})$, with $\Psi(z) = \alpha$. \square

1.2.2 Non-Symmetric Frobenius Algebras

We now look at the case when \mathcal{A} is not a symmetric Frobenius algebra. Since

$$\varepsilon : \mathcal{A} \rightarrow k$$

is non-degenerate, one can define an involution $f : \mathcal{A} \rightarrow \mathcal{A}$ such that $\varepsilon(ab) = \varepsilon(bf(a))$, $f^2(a) = a$, $f(ab) = f(a)f(b)$. Using this involution, one can state a duality very similar to the one we have derived in the case of the symmetric Frobenius algebras.

Given the involution f on \mathcal{A} , let us denote the "twisted" commutator of a and b by $[a, b]_t = ab - bf(a)$. Similarly, let $[\mathcal{A}, \mathcal{A}]_t$ be the subalgebra of \mathcal{A} spanned by elements $ab - bf(a)$. We also define the twisted center of \mathcal{A} as follows:

$$TZ(\mathcal{A}) = \{b \in \mathcal{A} \mid ba - f(a)b = 0 \text{ for all } b \in \mathcal{A}\}.$$

Note that if $b \in TZ(\mathcal{A})$ then $f(b) \in TZ(\mathcal{A})$ as well because

$$f(f(b)a) = bf(a) = at = f(f(a)f(b)),$$

which means that $f(b)a = f(a)f(b)$ since the involution f is bijective. Moreover, if $b \in TZ(\mathcal{A})$, we have

$$f(b)f(a) = f(ba) = f(f(a)b) = af(b),$$

such that for a general $t \in TZ(\mathcal{A})$, $at - tf(a) = 0$.

Claim 1.2. With the above notation,

$$Z(\mathcal{A}) \simeq (\mathcal{A}/[\mathcal{A}, \mathcal{A}]_t)^* \tag{1.1}$$

$$TZ(\mathcal{A}) \simeq (\mathcal{A}/[\mathcal{A}, \mathcal{A}])^*. \tag{1.2}$$

Proof. (1) The proof closely resembles the one for \mathcal{A} symmetric. Let

$$\begin{aligned}\Psi : Z(\mathcal{A}) &\rightarrow (\mathcal{A}/[\mathcal{A}, \mathcal{A}]_t)^* \\ z &\mapsto z\varepsilon : \mathcal{A} \rightarrow k; z\varepsilon(a) = \varepsilon(za).\end{aligned}$$

Ψ is well-defined:

$$z\varepsilon([a, b]_t) = z\varepsilon(ab - bf(a)) = \varepsilon(zab - zbf(a)) = \varepsilon(a(zb)) - \varepsilon((zb)f(a)) = 0$$

since z is central and $\varepsilon(ab) = \varepsilon(bf(a))$.

Ψ is injective: The proof is exactly as for \mathcal{A} symmetric.

Ψ is surjective: Let $\alpha \in (\mathcal{A}/[\mathcal{A}, \mathcal{A}]_t)^*$ and $\pi^* : (\mathcal{A}/[\mathcal{A}, \mathcal{A}]_t)^* \rightarrow \mathcal{A}^*$. Then $\pi^*(\alpha) = z\varepsilon$ for some $z \in \mathcal{A}$ and $z\varepsilon([a, b]_t) = 0$ for all $a, b \in \mathcal{A}$. We must verify that z is in the center $Z(\mathcal{A})$.

$$0 = z\varepsilon([a, b]_t) = \varepsilon(zab) - \varepsilon(zbf(a)) = \varepsilon(zab) - \varepsilon(azb) = \varepsilon([z, a]b).$$

Since this is true for all $b \in \mathcal{A}$ and since ε is non-degenerate, then $[z, a]$ must vanish for all $a \in \mathcal{A}$, which means that z is central.

(2) Let

$$\begin{aligned}\Psi : TZ(\mathcal{A}) &\rightarrow (\mathcal{A}/[\mathcal{A}, \mathcal{A}])^* \\ t &\mapsto t\varepsilon : \mathcal{A} \rightarrow k; t\varepsilon(a) = \varepsilon(ta).\end{aligned}$$

Ψ is well-defined:

$$c\varepsilon([a, b]) = \varepsilon(cab) - \varepsilon(cba) = \varepsilon(cab) - \varepsilon(baf(c)) = \varepsilon(cab) - \varepsilon(bf(c)f(a)) = \varepsilon(cab) - \varepsilon(bf(ca)) = 0.$$

Ψ is injective: The proof is as before.

Ψ is surjective: We have as before $c\varepsilon = \pi^*(\alpha)$ and we need to check that $c \in TZ(\mathcal{A})$.

$$0 = c\varepsilon([a, b]_t) = \varepsilon(cab) - \varepsilon(cba) = \varepsilon(cab) - \varepsilon(f(a)cb) = \varepsilon([t, a]_t b)$$

so that $[c, a]_t = 0$ and $c \in TZ(\mathcal{A})$. □

Note that $\mathcal{A}/[\mathcal{A}, \mathcal{A}]$ and $\mathcal{A}/[\mathcal{A}, \mathcal{A}]_t$ are no longer algebras, but vector spaces, both for $\mathcal{A} = \mathcal{NC}_n$ and $\mathcal{A} = \mathcal{H}_n$.

1.2.3 The Nilcoxeter and 0-Hecke Algebras as Non-Symmetric Frobenius Algebras

We can endow the 0-Hecke and Nilcoxeter algebras with a trace map, making them Frobenius algebras. Some of our constructions and arguments will be similar for both \mathcal{NC}_n and \mathcal{H}_n . In these cases we denote either of them by \mathcal{A}_n , where \mathcal{A}_n is generated by T_1, T_2, \dots, T_{n-1} , if we want to emphasize the dimension of our algebra, or simply by \mathcal{A} . Then $\varepsilon : \mathcal{A} \rightarrow k$ is defined as $\varepsilon(T_\sigma) = 1$ if $l(\sigma)$ is maximal (the maximal permutation has length $n(n-1)/2$ for the algebra on n strands) and $\varepsilon(T_\sigma) = 0$ otherwise. With a slight abuse of notation, we now refer to σ or even T_σ as maximal when $l(\sigma)$ is maximal. The maximal T_σ is called max_n .

Proposition 1.3. ε is a non-degenerate trace map and makes \mathcal{NC}_n and \mathcal{H}_n into Frobenius algebras.

Proof. ε non-degenerate means that for any $\sigma \in S_n$, there exist $\gamma, \gamma' \in S_n$ such that $T_\sigma T_\gamma = T_{\gamma'} T_\sigma = \max_n$. The properties of the length function on S_n and the multiplication rules derived in 1.1.1 and 1.1.2 imply that this is indeed the case. \square

The 0-Hecke and Nilcoxeter algebras are not symmetric. However, one can define an involution f on \mathcal{A} for both $\mathcal{A} = \mathcal{NC}_n$ and $\mathcal{A} = \mathcal{H}_n$:

$$\begin{aligned} f(T_i) &= T_{n-i}, 1 \leq i \leq n-1 \\ f(ab) &= f(a)f(b). \end{aligned}$$

It is clear the $\forall \sigma, f(T_\sigma) = T_\gamma$ for some γ , and that $l(\sigma) = l(\gamma)$. It immediately follows that σ is maximal if and only if γ is maximal, so that $\varepsilon(T_\sigma) = 0$ if and only if $\varepsilon(T_\gamma) = 0$. Hence, $\varepsilon(a) = \varepsilon(f(a))$ for all $a \in \mathcal{A}$.

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2 Topological Calculi of Diagrams

In order to determine the dimension of the center of our Frobenius algebra $\mathcal{A} = \mathcal{NC}_n$ or $\mathcal{A} = \mathcal{H}_n$, we make use of relation (1) of Claim 1.2 and determine the dimension of $\mathcal{A}/[\mathcal{A}, \mathcal{A}]_t$. A basis element of \mathcal{A} can be represented as a monotonic immersion of n unit intervals $[0, 1]$ in $\mathbb{R} \times [0, 1]$, subject to some relations. We can see this element as n strings each going from a top position $1 \leq i \leq n$ at 1 to a bottom position $1 \leq j \leq n$ at 0. A generator T_i of \mathcal{A} is represented by the crossing of the i and $i+1$ strings. The braid relations of \mathcal{A} allow us to change this diagram of strings using the third Reidemeister move ($T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$) and to move far away crossings up and down with respect to each other ($T_i T_j = T_j T_i$ for $|i-j| > 1$). We read diagrams from top to bottom.

2.1 Diagrams on a Flat Band

A diagram like the one described above, a monotonic immersion of n unit intervals $[0, 1]$ in $\mathbb{R} \times [0, 1]$, can be used to represent a basis element of the symmetric group ring $\mathbb{C}[S_n]$ if we impose the usual Reidemeister moves as relations. By changing the relations appropriately, it can also represent a basis element of \mathcal{NC}_n or \mathcal{H}_n . To $s_i \in S_n$ or $T_i \in \mathcal{A}$ we assign a crossing of the i and $i+1$ strands, to a product $s_i s_j$ or $T_i T_j$ we assign a crossing of the i and $i+1$ strands followed by a crossing of the j and $j+1$ strands and to a linear combination

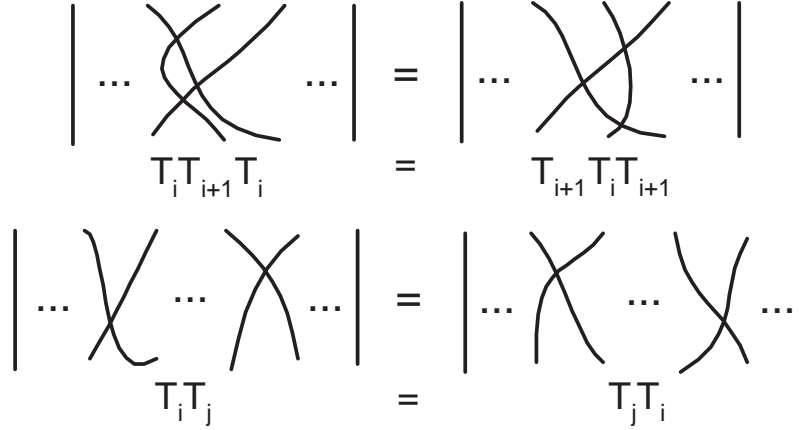


Figure 1: These diagrams show the graphical relations common to $\mathbb{C}[\mathcal{S}_n]$, \mathcal{NC}_n and \mathcal{H}_n . The relation used to handle T_i^2 is what distinguishes these algebras.

of basis elements we assign a formal sum of diagrams. The relations imposed are:
for a basis element of $\mathbb{C}[\mathcal{S}_n]$:

1. $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$
2. $s_i s_j = s_j s_i$ for $|i - j| > 1$
3. $s_i^2 = 1$

for a basis element of \mathcal{NC}_n :

1. $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$
2. $T_i T_j = T_j T_i$ for $|i - j| > 1$
3. $T_i^2 = 0$

and for a basis element of \mathcal{H}_n :

1. $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$
2. $T_i T_j = T_j T_i$ for $|i - j| > 1$
3. $T_i^2 = T_i$.

Right multiplication of two basis elements b_1, b_2 then becomes the juxtaposition of the diagram for b_2 under the diagram for b_1 , then rescaled to be in $\mathbb{R} \times [0, 1]$ as shown in Figure 3.

To each word $w = s_{i_1} \dots s_{i_k}$ in \mathcal{S}_n there is assigned an element $T_w = T_{i_1} \dots T_{i_k}$ of \mathcal{A} and a diagram D . In the Nilcoxeter algebra case, if w is not reduced, then, $T_w = 0$ and $D = 0$.

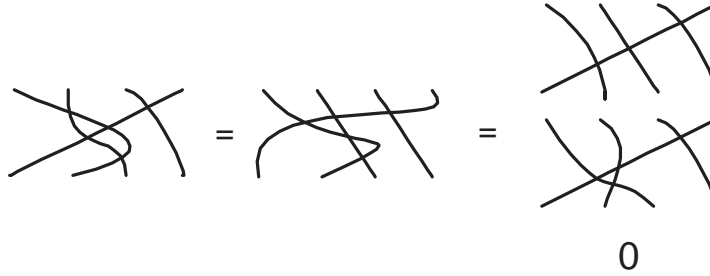


Figure 2: This first diagram is equivalent to three distinct diagrams in $\mathbb{C}[\mathcal{S}_n]$, \mathcal{H}_n and \mathcal{NC}_n respectively.



Figure 3: Multiplication of basis elements corresponds to concatenation of diagrams.

In the 0-Hecke algebra case, $T_w = T_{w'}$ for some reduced word w' and likewise, $D = D'$ for some reduced diagram D' . Note that in general w and w' are not equivalent words in \mathcal{S}_n . Moreover, because a reduced word in \mathcal{S}_n can be related to any other equivalent reduced word through the braid relations alone, diagrams in \mathcal{NC}_n or \mathcal{H}_n corresponding to reduced words equivalent in \mathcal{S}_n are equivalent. Conversely, if two reduced diagrams are equivalent, then the reduced words to which they correspond are also equivalent in \mathcal{S}_n .

2.2 Diagrams on a Cylinder: The Quotients of the Nilcoxeter and 0-Hecke Algebras by Their Commutator Subgroups

Looking at diagrams on $\mathbb{R} \times S^1$ instead of $\mathbb{R} \times [0, 1]$ imposes the new relation $ab = ba$ on \mathcal{A} as shown in Figure 4. This is equivalent to working in $\mathcal{A}/[\mathcal{A}, \mathcal{A}]$.

Figure 5 exemplifies some of the consequences of this new relation for $\mathcal{A} = \mathcal{NC}_n$. Recall that $\mathcal{A}/[\mathcal{A}, \mathcal{A}]$ is no longer an algebra but a vector space, both in the case $\mathcal{A} = \mathcal{NC}_n$ and $\mathcal{A} = \mathcal{H}_n$.

2.3 Diagrams on the Möbius band: The Quotients of the Nilcoxeter and 0-Hecke Algebras by Their Twisted Commutator Subgroups

A basis element of $\mathcal{A}/[\mathcal{A}, \mathcal{A}]_t$ can be represented as a diagram of n strings living on a Möbius band. On the Möbius band, a crossing T_i , if last in the diagram, can be pushed around the band to become a crossing T_{n-i} first in the diagram. Figure 6 shows how this

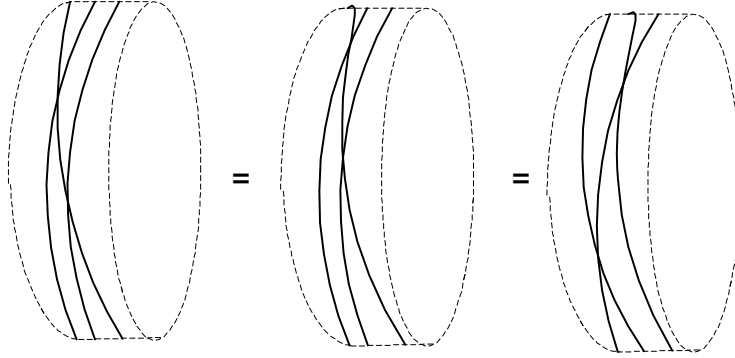


Figure 4: Being on a cylinder imposes the relation $ab = ba$ since the top crossing can be carried around the cylinder to become the bottom crossing.



Figure 5: These first two diagrams represent basis elements equivalent in $\mathcal{A}/[\mathcal{A}, \mathcal{A}]$ but not in \mathcal{A} for $\mathcal{A} = \mathcal{NC}_n$. The third one represent the zero element in $\mathcal{A}/[\mathcal{A}, \mathcal{A}]$ for $\mathcal{A} = \mathcal{NC}_n$. This is because of the new relation $ab = ba$ which is imposed by working on a cylinder instead of a flat band.

can be done. Thus, $TT_i = T_{n-i}T$ as desired in $\mathcal{A}/[\mathcal{A}, \mathcal{A}]_t$. Indeed, on the Möbius band, this new relation corresponds exactly to the involution used to define the relation on $\mathcal{A}/[\mathcal{A}, \mathcal{A}]_t$. We typically represent these diagrams on a flat band, but one should mentally identify the top and bottom of the band in opposite directions so as to form a Möbius band. Figure 7 gives an example of Nilcoxeter diagrams which are equivalent on the Möbius band but not on a flat band or on a cylinder. The elements they represent are therefore identified in $\mathcal{A}/[\mathcal{A}, \mathcal{A}]_t$.

With the goal of understanding the center of \mathcal{A} , we classify these diagrams on the Möbius band. The cases $\mathcal{A} = \mathcal{NC}_n$ and $\mathcal{A} = \mathcal{H}_n$ are very similar. We therefore carry out this classification in detail for $\mathcal{A} = \mathcal{NC}_n$ and later give a summary of the corresponding results for $\mathcal{A} = \mathcal{H}_n$.

3 Nilcoxeter Diagrams on the Möbius band

To begin our study of diagrams on the Möbius band, we fix some notation. First note that on the Möbius band, as well as on the cylinder, these n strings could actually be any number $c \leq n$ of immersed circles. Each circle is called a component of the diagram and the number of times a single immersed circle goes around the Möbius band, i.e. the

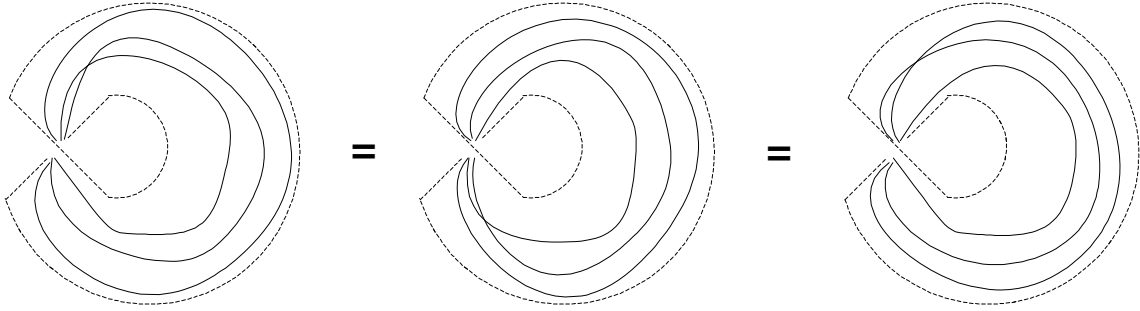


Figure 6: Being on the Möbius band imposes the new relation $ab = bf(a)$ with $f(T_i) = T_{n-i}$ since a crossing T_i can be dragged along the Möbius band to become T_{n-i} .

degree of that immersion, is called the thickness of that component. The sum of the individual thicknesses of the components of a diagram is n , the thickness of the diagram. Figure 8 shows four diagrams of thickness four that decompose into components of different thicknesses. In general, one can see that a diagram of thickness $2k$ with no crossing must consist of k components each of thickness 2; one of the components having both top and bottom positions 1 and n , another positions 2 and $n - 1$, etc. The diagram corresponding to the image of max_n in $\mathcal{A}/[\mathcal{A}, \mathcal{A}]_t$, the diagram with the maximum number of crossings, must consist of n components each of thickness 1; one component having top position 1 and bottom position n , another top position 2 and bottom position $n - 1$, etc.

We now want to introduce the notion of a path p . A path is defined in a specific component C of a diagram D . If C has thickness m and D has thickness n , we define p as a sequence $(p(0), p(1), \dots, p(m-1))$ where $1 \leq p(i) \leq n$, $p(i) \in \{\text{top position of a strand which belongs to } C\}$ is the top position of the strand of C after going around the Möbius band i times starting at $p(0)$. We always consider this sequence $\text{mod}(n)$ and all statements made about it should be understood as such. Figure 9 gives an example of a path for the central component of thickness four of this composite diagram of total thickness six. The starting position of this path is $p(0) = 3$.

For each component C , this sequence is determined by D once $p(0)$ is chosen. Hence, two paths p, p' in the same component can only differ by an integer $l \leq m$, the lag of one path behind the other such that $p(i) = p'(i + l)$.

3.1 Prime Diagrams

A prime diagram is a diagram consisting of a single component. We will see that there is a unique prime diagram of thickness n for each n .

3.1.1 The Minimum Number of Crossings of a Prime Diagram

First, let us look at the minimal number of crossings such a diagram must have. If, instead of being on the Möbius band, we were on a simple cylinder $S^1 \times [1, n]$, the minimal condition

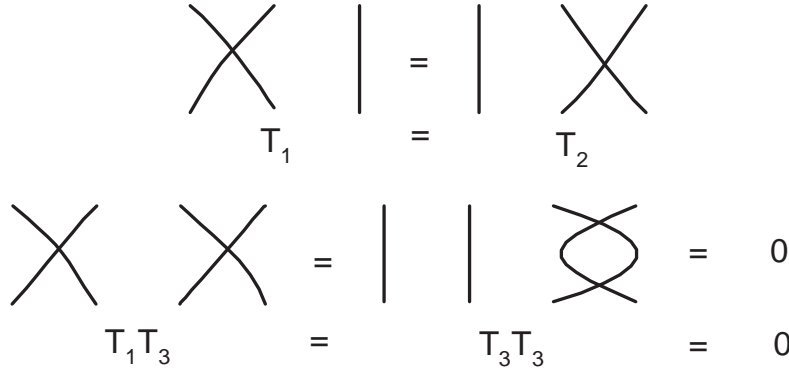


Figure 7: The first two diagrams of this Figure are representations of the same element in $\mathcal{A}/[\mathcal{A}, \mathcal{A}]_t$. On the Möbius band, if the left crossing is pushed once around, it becomes the right crossing. This corresponds to the identification of diagrams in \mathcal{A} under the involution $f(T_i) = T_{n-i}$. The second diagram is 0 in $\mathcal{A}/[\mathcal{A}, \mathcal{A}]_t$ for $\mathcal{A} = \mathcal{NC}_n$.

for the n strands to be connected, to be part of the same component, would be clear: the diagram would have to correspond to an n -cycle in \mathcal{S}_n , in particular, it would need to have at least the $n - 1$ possible crossings T_1, \dots, T_{n-1} . In fact, it needs to have exactly each possible crossing once since on the cylinder, two strings are part of the same component if and only if they cross exactly once. Similarly, since the string with top position i and the one with bottom position $n - i + 1$ are already part of the same component in a diagram on the Möbius band, one can quickly see that on the Möbius band, a diagram of thickness n needs one of T_i or T_{n-i} for each $1 \leq i \leq \lfloor (n - 1)/2 \rfloor$ to be connected. The minimum number of crossings required is therefore $\lfloor (n - 1)/2 \rfloor$.

3.1.2 The Exact Number of Crossings of a Prime Diagram

We need the following result to determine the maximum number of crossings that a non-zero prime diagram can have.

Proposition 3.1. *A diagram D on the Möbius band is zero if and only if there are more than one crossing between two fixed paths.*

Proof. The first implication is clear: if there are two consecutive crossings T_i for some i in D , then these two crossings must occur between the same two paths. Conversely, given two paths p and p' , one can see them as being both “on top” of the other strands of D for each i . One can then slide a crossing occurring between these two paths around the Möbius band to the next or previous i . The only obstacle to doing this is if there is already a crossing between these paths at this i , in which case we have two consecutive crossings and $D = 0$. If there are two crossings between these two paths occurring at different i , one can

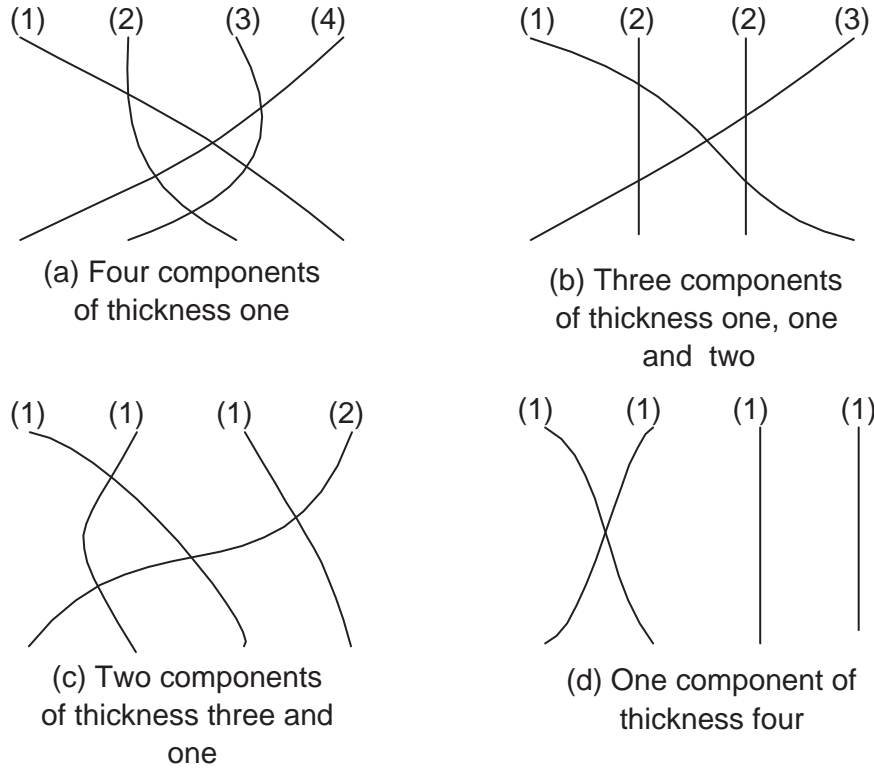


Figure 8: These four non-zero diagrams are all of total thickness four but decompose into different prime components.

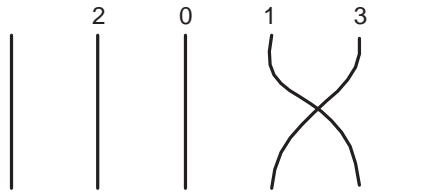


Figure 9: The central component of thickness four of this diagram was here marked with the path $(p(0), p(1), p(2), p(3)) = (3, 4, 2, 5)$. The other component has thickness two.

then move one of them around the Möbius band until they are consecutive and annihilate the diagram. Figure 10 shows how this is done for a specific example.

□

We have already shown that in order for D to be prime, it must have at least $\lfloor (n-1)/2 \rfloor$ crossings. We now check that this is actually the maximum number of crossings a prime diagram of thickness n can have if it is to be non-zero.

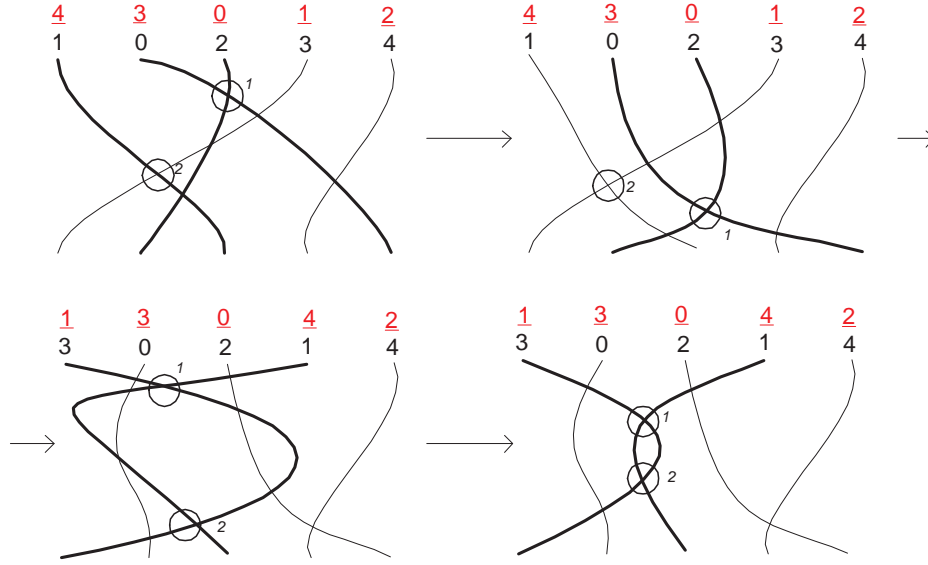


Figure 10: Two paths and two crossings have been marked on this Figure. Crossing 1 occurs at $i = 0$ and crossing 2 at $i = 1$. The bold strands are considered to be "on top" of the others, allowing us to see easily how to move crossings between them. One can first slide crossing 1 down the diagram. Then it can be pushed around the Möbius band to $i = 1$ so that the two crossings now both occur at $i = 1$. They become consecutive and D is seen to be zero. Note that these two paths actually cross a third time originally at $i = 3$.

Proposition 3.2. A prime diagram of thickness n has exactly $\lfloor (n-1)/2 \rfloor$ crossings

Proof. Let us fix a component C . We want to look at (unordered) pairs of paths in C , but some pairs are redundant. For instance, in Figure 9, we would like the two pairs of paths $(p = (3, 4, 2, 5), p' = (2, 5, 3, 4))$ and $(q = (4, 2, 5, 3), q' = (5, 3, 2, 4))$ to be equivalent. We therefore consider pairs of paths modulo relative starting points, that is we impose the relation

$$(p, p') \equiv (q, q') \Leftrightarrow (p(i), p'(i)) = (q(i+c), q'(i+c))$$

for some integer c (remember that we are working $\text{mod}(m)$). Note that if p' lags on p by l , then p lags on p' by $m-l$. Since we are looking at unordered pairs and since any two pairs with the same lag are equivalent by the above relation, any pair of paths (p, p') such that $p(i) = p'(i+l)$ is equivalent to any pair (q, q') such that $q(i) = q'(i+n-l)$ since then $q'(i) = q(i+l)$. The pairs of paths modulo relative starting points are therefore indexed by an integer $1 \leq l \leq \lfloor m/2 \rfloor$. In our example of Figure 9, we thus have a total of 2 classes of pairs of paths, those which have a lag of 1 or equivalently of 3 such as $(p = (1, 3, 4, 2), p' = (2, 4, 1, 3))$, and those which have a lag of 2 such as $(q = (1, 3, 4, 2), q' = (3, 1, 2, 4))$.

Remember that when the same pair of paths intersects twice, the diagram is zero. Moreover, a crossing belongs to a pair of paths if and only if it also belongs to all equivalent

pairs. Thus, there can be only one crossing per equivalence class of pairs of paths for the component to be non-zero. Since there are $\lfloor m/2 \rfloor$ equivalence classes of pairs of paths, there can be a maximum of $\lfloor m/2 \rfloor$ crossings within a component of thickness m for the diagram D to be non-zero.

In this section, we are concerned with prime diagrams, so now let $D=C$. For a prime diagram of odd thickness n , the maximum number of crossings we have just derived $\lfloor n/2 \rfloor = \lfloor (n-1)/2 \rfloor$, the minimum number of crossings previously derived, and the Proposition is proved. For D of even thickness n , we will show that D cannot actually realize this maximum of $\lfloor n/2 \rfloor$ crossings; it has either no more than $n/2 - 1 = \lfloor (n-1)/2 \rfloor$ crossings or is zero, which completes the proof of Proposition 3.2.

In fact, we show that a non-zero even prime diagram cannot have a middle crossing $T_{n/2}$. First, note that the paths p and p' , $p(0) < p'(0)$ never intersect iff

$$p(2k) < p'(2k)$$

and

$$p(2k+1) > p'(2k+1)$$

for all k (again, we consider both $2k$ and $2k+1$ as integers $\text{mod}(n)$). Now, assume that D contains no middle crossing $T_{n/2}$ and consider the paths (p, p') ; $p(0) = n/2$, $p'(0) = n/2 + 1$. Since no strand ever crosses the middle point of the diagram, we have

$$p(2k) \leq n/2, p'(2k) \geq n/2 + 1$$

and

$$p(2k+1) \geq n/2 + 1, p'(2k+1) \leq n/2$$

such that $p(2k) < p'(2k)$ and $p(2k+1) > p'(2k+1)$ and p, p' do not intersect. Having at least one class of pairs of paths which do not intersect, such a diagram D has a maximum of $n/2 - 1 = \lfloor (n-1)/2 \rfloor$ crossings. Assume then that D does contain a middle crossing $T_{n/2}$. Without loss of generality, assume that $T_{n/2}$ is the top crossing of the diagram and consider the same paths (p, p') ; $p(0) = n/2, p'(0) = n/2 + 1$. Now, since these paths cross immediately, $p(1) \leq n/2 < n/2 + 1 \leq p'(1)$. Suppose that these paths do not intersect again, then

$$p(2k) > p'(2k)$$

and

$$p(2k-1) < p'(2k-1)$$

for $k > 0$. But we know that

$$n/2 = p(0) = p(n) = p(2(n/2)) < p'(2(n/2)) = p'(n) = p'(0) = n/2 + 1,$$

which is a contradiction. Therefore, (p, p') must intersect more than once and D is zero. \square

3.1.3 The Crossings of a Prime Diagram

We have now shown that a prime diagram of thickness n must have exactly $\lfloor (n-1)/2 \rfloor$ crossings. Moreover, we know that these crossings are exactly T_i or T_{n-i} for each $1 \leq i \leq \lfloor (n-1)/2 \rfloor$. In any case, each T_i appears at most once, which allows us to push all of the crossings to the left side of the diagram and thus assume that the $\lfloor n/2 \rfloor$ crossings are exactly T_i for $1 \leq i \leq \lfloor (n-1)/2 \rfloor$. Indeed, to see this first assume inductively that all T_j , for $j > n-i$, have already been pushed to the left side. We can do this since there is at most one T_{n-2} crossing that could prevent T_{n-1} from being moved. If there is a T_{n-2} crossing above or below T_{n-1} , the latter is free to be pushed in the other direction and if there is no T_{n-2} then the T_{n-1} crossing can be pushed either up or down the Möbius band to become T_1 . Consider now what could block T_{n-i} . Everything to its right has already been pushed, so only T_{n-i-1} above or below it can block it. It is therefore free to be pushed in the other direction. Therefore, from now on when we think of a prime diagram, we always represent it with its crossings being $T_1, T_2, \dots, T_{\lfloor (n-1)/2 \rfloor}$ in some order. It remains to show that these $\lfloor (n-1)/2 \rfloor!$ diagrams are in fact all equal in $\mathcal{A}/[\mathcal{A}, \mathcal{A}]_t$.

3.1.4 Uniqueness of Prime Diagrams

Lemma 3.3. $T_1 T_2 \dots T_{\lfloor (n-1)/2 \rfloor} = T_{\sigma(1)} T_{\sigma(2)} \dots T_{\sigma(\lfloor (n-1)/2 \rfloor)} \in \mathcal{A}/[\mathcal{A}, \mathcal{A}]_t$ for all $\sigma \in S_{\lfloor (n-1)/2 \rfloor}$

Proof. First, let us look at the case when σ is a cyclic permutation. Fix $m = \lfloor (n-1)/2 \rfloor$. Now,

$$\mathbf{T}_1(T_2 \dots T_m) = (T_2 \dots T_m)\mathbf{T}_{n-1} = \mathbf{T}_{n-1}(T_2 \dots T_m) = (T_2 \dots T_m)\mathbf{T}_{n-(n-1)=1}$$

since $|(n-1) - i| > 1$ for $2 \leq i \leq m$. More generally, if we have any diagram $D = T_{\sigma'(1)} T_{\sigma'(2)} \dots T_{\sigma'(m)}$ for $\sigma' \in S_m$, then

$$\begin{aligned} \mathbf{T}_{\sigma'(1)}(T_{\sigma'(2)} \dots T_{\sigma'(m)}) &= (T_{\sigma'(2)} \dots T_{\sigma'(m)})\mathbf{T}_{n-\sigma'(1)} = \\ \mathbf{T}_{n-\sigma'(1)}(T_{\sigma'(2)} \dots T_{\sigma'(m)}) &= (T_{\sigma'(2)} \dots T_{\sigma'(m)})\mathbf{T}_{\sigma'(1)} \quad (*) \end{aligned}$$

because in fact $n - \sigma'(1) > m + 1$ so that $|(n - \sigma'(1)) - i| > 1$ for $1 \leq i \leq m$ unless n is odd and $\sigma'(1) = m$. Then $n - \sigma'(1) = m + 1$, but we still have that $|(m + 1) - \sigma'(j)| > 1$ for $2 \leq j \leq m$ since $\sigma'(j) \neq m$ for $j \neq 1$ so that (*) still holds. Diagrammatically, we have taken the top crossing and pushed it onto the right side of the diagram, where it is free to be pushed to the top again. We can then push it back to the bottom left side of the diagram. Hence the Lemma holds for σ a cyclic permutation. Figure 11 shows how this is done for a simple example.

Now let us compose a permutation σ' with a transposition $(i \ i+1)$. Since we already know that this diagram is invariant under cyclic permutations of its crossings, we can actually assume that $i = 1$. If $\sigma'(2) \neq \sigma'(1) \pm 1$, $\sigma'(1)$ and $\sigma'(2)$ commute and we are done.

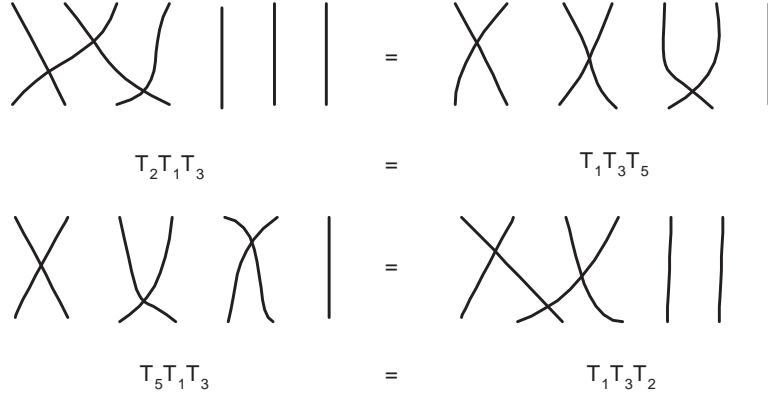


Figure 11: This Figure shows how one can cyclically permute the crossings of a non-zero prime diagram.

Without loss of generality, assume therefore that $\sigma'(2) = \sigma'(1) + 1$. Let k be such that $\sigma'(k) = \sigma'(2) + 1$. So

$$\begin{aligned}
& \mathbf{T}_{\sigma'(2)} \mathbf{T}_{\sigma'(1)} (\dots T_{\sigma'(k)} \dots T_{\sigma'(m)}) = \\
& (T_{n-\sigma'(k)} \dots T_{n-\sigma'(m)}) (\mathbf{T}_{\sigma'(2)} \mathbf{T}_{\sigma'(1)} (\dots T_{\sigma'(k-1)})) = \\
& \mathbf{T}_{\sigma'(2)} (T_{n-\sigma'(k)} \dots T_{n-\sigma'(m)} \mathbf{T}_{\sigma'(1)} \dots T_{\sigma'(k-1)}) = \\
& (T_{n-\sigma'(k)} \dots T_{n-\sigma'(m)} \mathbf{T}_{\sigma'(1)} \dots T_{\sigma'(k-1)}) \mathbf{T}_{\sigma'(2)} = \\
& (T_{n-\sigma'(k)} \dots T_{n-\sigma'(m)}) \mathbf{T}_{\sigma'(1)} \mathbf{T}_{\sigma'(2)} (\dots T_{\sigma'(k-1)}) = \\
& \mathbf{T}_{\sigma'(1)} \mathbf{T}_{\sigma'(2)} (\dots T_{\sigma'(k)} \dots T_{\sigma'(m)})
\end{aligned}$$

using the fact that it is invariant under cyclic permutations. Thus this diagram is in fact invariant under all permutations $\sigma \in S_{\lfloor (n-1)/2 \rfloor}$. \square

This Lemma now allows us to conclude that there is exactly one prime diagram of thickness n for each n . This diagram can be represented as $D = T_1 T_2 \dots T_{\lfloor (n-1)/2 \rfloor}$.

3.2 Composite Diagrams

We now need to see how prime diagrams can be combined into non-zero composite diagrams. At first glance, there seems to be a number of ways to combine diagrams. Observe that, given two diagram D and D' with the same prime components C_1, \dots, C_k , it is always possible to go from D to D' via an isotopy. Indeed, given that there is no under and overcrossings, one can first consider that C_1 is on top and place it where one wants, then do the same with C_2 and thus simply rearrange the components of D as they are in D' . This isotopy, however, might very well go through zero in $\mathcal{A}/[\mathcal{A}, \mathcal{A}]_t$ if $\mathcal{A} = \mathcal{NC}_n$. In fact, we know that if the second Reidemeister move is used, then it does go through zero, and we also know that this move is the only one which changes the number of crossings.

Therefore, given isotopic diagrams D and D' , if D has more crossings than D' , then $D = 0$. Thus all non-zero diagrams in $\mathcal{A}/[\mathcal{A}, \mathcal{A}]_t$ with the same prime components C_1, \dots, C_k have the same number of crossings and that number is the minimum of the isotopy class. Note that such a minimum must exist and that the corresponding diagram, having no two paths crossing more than once since it is minimal, is non-zero.

Note also that the relations on these diagrams being local, any relation on a prime component continue to hold when the component is combined in a composite diagram.

3.2.1 Combining Even Prime Diagrams

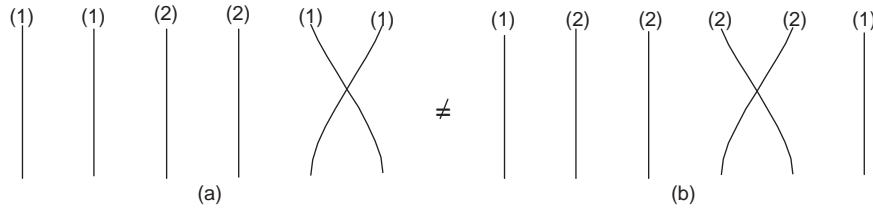


Figure 12: These two diagrams are both non-zero since they are minimal and their decomposition into prime components is the same. However, they do not represent the same element of $\mathcal{A}/[\mathcal{A}, \mathcal{A}]_t$. This shows that prime component decompositions does not quite determine unique elements in $\mathcal{A}/[\mathcal{A}, \mathcal{A}]_t$.

Consider a diagram D with even prime components C_1, C_2 of respective thicknesses $2k_1, 2k_2$. Since even prime diagrams have no middle crossings, we can combine C_1 and C_2 by either nesting C_2 inside C_1 , which means having C_1 at positions $1, 2, \dots, k_1, k_1 + 2k_2 + 1, k_1 + 2k_2 + 2, \dots, 2k_1 + 2k_2$ both on top and bottom and C_2 at positions $k_1 + 1, k_1 + 2, \dots, k_1 + 2k_2$ or vice versa. Neither of these two arrangements will add crossings between C_1 and C_2 : if C_1 has $k_1 - 1$ crossings and C_2 has $k_2 - 1$ crossings and D has $k_1 + k_2 - 2$ crossings in either arrangement, which are therefore both minimal. Let D be the diagram with C_1 in the center and D' the one with C_2 in the center. Then an isotopy from D to D' requires the second Reidemeister moves which creates double crossings and the isotopy therefore goes through zero. Hence, D and D' are distinct non-zero diagrams in $\mathcal{A}/[\mathcal{A}, \mathcal{A}]_t$.

3.2.2 Combining Odd and Even Components

Next, given a component K (prime or composite) of total odd thickness with k crossings, and even prime components C_1, \dots, C_m , with n_1, \dots, n_m crossings respectively, the minimal arrangements have no intercomponent crossings: they have $k + n_1 + \dots + n_m$ crossings. These arrangements all involve K in the center, disjoint from the even components, which are split each in the middle around K . If C_1, \dots, C_m are all distinct, then there are $m!$ arrangements of this type. By the same argument as in the case of two even diagrams, there is no isotopy between these arrangements that does not require the second Reidemeister move, so these

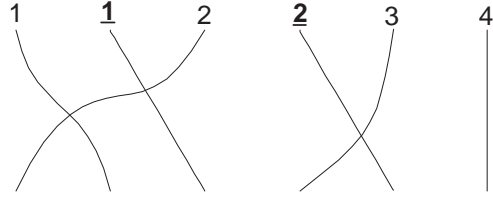


Figure 13: This is a non minimal arrangement of two even prime components. The strands are numbered with their positions in their respective prime components to show their assignment functions f_{C_i} more clearly. The inter-component crossings can be avoided by splitting one of the components in two and placing the other in the center. This diagram, not being minimal, is zero.

$m!$ diagrams are in fact distinct. Figure 14 is an example of such a minimal arrangement. Since K has odd thickness and thus cannot be split in its middle, if it is not in the center, then it is also not disjoint from the even components. The corresponding diagram is then not minimal and is therefore zero.

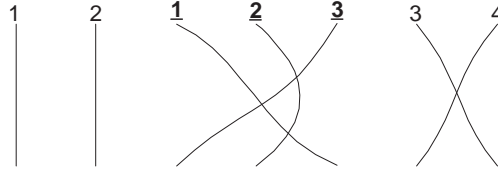


Figure 14: This is a minimal diagram with one odd (composite) and one even prime component. The strands are numbered with their positions in their respective prime components to show their assignment functions f_{C_i} more clearly. Minimal arrangements of this kind always involve placing the odd component in the center so as to not create inter-component crossings.

3.2.3 Combining Odd Prime Diagrams

Let us now see how to combine two odd diagrams C_1, C_2 of thicknesses $2k_1 + 1, 2k_2 + 1$ respectively into a diagram D . To try to find a combination, one can start by numbering $2k_1 + 1$ of the $2k_1 + 2k_2 + 2 := n$ top positions of D with numbers $1, \dots, 2k_1 + 1$ in a certain color. These would be the top position for the strands of C_1 . First note that once these top positions for C_1 are chosen, the bottom ones are determined. Indeed, if the number i was assigned to the top position j of D , then the number $n - i$ is assigned to the bottom position $n - j + 1$. We call this assignment from $\{1, 2, \dots, 2k_1 + 1\} \hookrightarrow \{1, 2, \dots, n\}$, f_{C_1} . There seems to be many ways one can choose the positions of C_1 and C_2 in D , but we will see that there is in fact a unique choice that makes D non-zero.

Lemma 3.4. *There is a unique non-zero diagram D with prime components C_1, \dots, C_m of respective thicknesses $2k_1 + 1, \dots, 2k_m + 1$.*

Proof. We describe an arrangement and then argue that it is the unique one with the minimum number of crossings. Without loss of generality, assume that

$$2k_1 + 1 \leq 2k_2 + 1 \leq \dots \leq 2k_m + 1.$$

Let $n := 2k_1 + 2k_2 + \dots + 2k_m + m$ be the total thickness of D . We build the diagram component by component. Let D_i be the diagram of thickness $n_i := 2k_1 + 2k_2 + \dots + 2k_i + i$ with prime components C_1, \dots, C_i obtained after the i^{th} step of this procedure. The following assignment functions are the ones yielding the unique minimal diagram D_{i+1} :

$$f_{C_{i+1}}(j) = \begin{cases} j & j \leq k_{i+1} \\ j + n_i + k_{i+1} & j > k_{i+1} \end{cases}$$

and

$$f_{D_i}(l) = l + k_{i+1},$$

which means that we keep the previous composite D_i in the center and give C_{i+1} the edge positions.

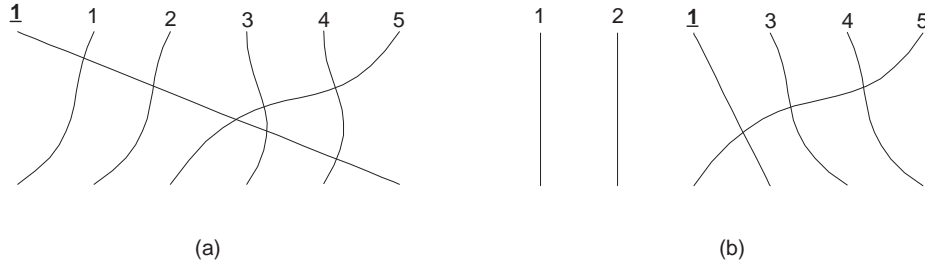


Figure 15: The strands of these two diagrams are numbered with their positions in their respective prime components to show their assignment functions f_{C_i} more clearly. Diagram (a) is non-minimal because the thickest odd component was placed in the center of the diagram, creating 1×5 inter-component crossings, while diagram (b), by placing the thickest component on the outside, minimized inter-component crossings to 1×1 .

First, we want to minimize the number of extra crossings of a component with itself. If possible, it is clear that a minimal diagram is one where the placement of each component does not add crossings between strands of the same component. If f is not monotonic, extra crossings might be created, while none are added if f is monotonic. Therefore, monotonic assignment functions create minimal diagrams. Note that for each component of thickness m , if $f' = m + 1 - f$ then f and f' are equivalent on the Möbius band. Indeed, diagrams which are reflections of each other are seen to be the same by rotating the Möbius band. We therefore only consider strictly increasing assignment functions f . Moreover, all

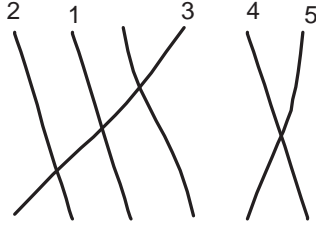


Figure 16: The assignment function f_{C_i} shown by the numbered strands is not monotonic and does not preserve the number of crossings of that prime component of thickness five. This diagram is therefore zero.

assignment functions f_{C_i} for C_i a prime component which do not increase the number of crossings within C_i itself are equivalent since all diagrams of C_i with the minimal number of crossings are equivalent.

We also want to minimize crossings between D_i and C_{i+1} , which means that we want to avoid alternating positions, ie. situations like

$$f_{D_i}(j) < f_{C_{i+1}}(k) < f_{D_i}(l)$$

which add intercomponent crossings. For a component C of thickness m in a diagram of total thickness n , the bottom j position is the same as the top j position whenever

$$f(j) = n + 1 - f(m + 1 - j).$$

When this is not true, we have additional intercomponent crossings. Therefore, minimizing the crossings means that we want one component in the center and one on the outside of the diagram, each symmetric in their assignment.

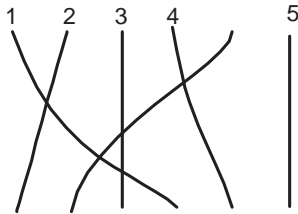


Figure 17: The assignment functions f_{C_i} shown by the numbered strands is not symmetric and does not minimize the number of intercomponent crossings. This diagram is therefore zero.

Inductively, we see that in any diagram, there is one strand per odd component crossing the center line of the diagram. Therefore, if D_i is placed as the outside component in D_{i+1} , there are $i \times (2k_{i+1} + 1)$ crossings between D_i and C_{i+1} , since D_i is composed of i odd prime

components and one strand for each of these prime components will cross the $2k_{i+1} + 1$ strands of C_i . Now, an odd component cannot really be split in two and placed on the outside because of the symmetry requirement. Placing it on the outside therefore implies taking the first and last k position, both top and bottom to minimize crossings. The remaining position has bottom position assignment $n_{i+1} + 1 - j$ if it has top assignment j . C_{i+1} being an odd prime component, we know that, in the standard representation of this diagram, the strand having top position $k_{i+1} + 1$ has bottom position 1 in C_{i+1} and the one having bottom position $k_{i+1} + 1$ has top position k_{i+1} . This means that in D_{i+1} , if $f_{C_{i+1}}(k_{i+1}) = j$, the strand having top position $k_{i+1} + 1$ in C_{i+1} will cross D_i $j - k_{i+1}$ times and the one with bottom position $k_{i+1} + 1$ in C_{i+1} will cross D_i $n_{i+1} + 1 - j - (k_{i+1} + 1)$ times for a total of

$$j - k_{i+1} + n_{i+1} + 1 - j - (k_{i+1} + 1) = n_{i+1} - (2k_{i+1} + 1) = n_i$$

times. Now, $i \times (2k_{i+1} + 1) > n_i$ since we have ordered the components such that

$$k_1 \leq k_2 \leq \dots$$

Therefore, placing the (thickest) prime odd component C_{i+1} as the outside component always yields a minimal diagram. It remains to see whether the diagram is invariant under the choice of $f_{C_{i+1}}(k_{i+1}) = j$, the position in D_{i+1} of the middle position of C_{i+1} . This can be seen by pushing the crossing T_{j-1} around the Möbius band so that it becomes $T_{n_{i+1}+1-j-1}$. The diagram thus becomes exactly the diagram with $f_{C_{i+1}}(k_{i+1}) = j - 1$. Doing the opposite, we get the diagram with $f_{C_{i+1}}(k_{i+1}) = j + 1$, and so we see that D_{i+1} is invariant under the choice of j . There is therefore a unique diagram D on the Möbius band with prime components C_1, \dots, C_m . \square

This completes the classification of Nilcoxeter diagrams on the Möbius band. We have learned that there is a unique prime diagram of thickness n for each n , that there is a unique diagram with only odd prime components C_1, \dots, C_m and that in general, the number of arrangements depends on the number of even prime components of each thickness.

4 The Center of the Nilcoxeter Algebra

4.1 The Dimension of the Center

We have now enumerated all the possible arrangements of prime components into composite diagrams. Given even prime components C_1, \dots, C_m , i_j of them with thickness $2j$ and odd prime components C'_1, \dots, C'_m , one can make exactly $\frac{m!}{\prod_j (i_j!)}$ distinct non-zero diagrams D with this prime decomposition. We can now count the number of non-zero diagrams of thickness n in $\mathcal{A}/[\mathcal{A}, \mathcal{A}]_t$, each of which is a basis element of $\mathcal{A}/[\mathcal{A}, \mathcal{A}]_t$. This number is also the dimension of the center Z of \mathcal{A} :

$$\dim(Z) = \sum_{\lambda \vdash n} \frac{n_\lambda!}{m_\lambda!},$$

where n_λ is the number of even parts in λ and $m_\lambda = \prod_j (i_j!)$.

For instance, for $n = 3$ we have partitions

$$(1, 1, 1), (1, 2), (3)$$

and the formula gives us

$$\dim(Z) = 0!/0! + 1!/1! + 0!/0! = 3.$$

A more interesting case is $n = 6$ where we have partitions

$$(1, 1, 1, 1, 1, 1), (1, 1, 1, 1, 2), (1, 1, 1, 3), (1, 1, 2, 2), (1, 1, 4), (1, 5), (1, 2, 3), (2, 2, 2), (2, 4), (3, 3), (6).$$

The formula then gives us

$$\dim(Z) =$$

$$0!/0! + 1!/1! + 0!/0! + 2!/2! = 1!/1! + 0!/0! + 1!/1! + 3!/3! + 2!/1!1! + 0!/0! + 1!/1! = 12.$$

Here the partition $(2, 4)$ gives us two distinct diagrams.

4.2 A Basis for the Center

We now know the dimension of the center of \mathcal{NC}_n , but we would also like to have a basis for this center. Recall that we used the fact that

$$Z(\mathcal{A}) \simeq \mathcal{A}/[\mathcal{A}, \mathcal{A}]_t^*$$

and that this isomorphism is given by the trace map ε . Therefore, we just need to find the dual of the diagrams we counted in the previous section. First, given an element $T_\alpha \in \mathcal{A}$, we call T_β complementary to T_α if $\varepsilon(T_\alpha T_\beta) = 1$, $\alpha, \beta \in \mathcal{S}_n$. Note that for $\mathcal{A} = \mathcal{NC}_n$, this is equivalent to $\alpha\beta = \max_n$, which also means that there is a unique complementary T_β for each T_α .

Now, we need to find the element of $Z(\mathcal{A})$ corresponding to a basis element of $\mathcal{A}/[\mathcal{A}, \mathcal{A}]_t$. Since the distinct diagrams of thickness n formed a basis for the vector space $\mathcal{A}/[\mathcal{A}, \mathcal{A}]_t$, their corresponding elements in $Z(\mathcal{A})$ are also linearly independent and therefore form a linear basis for $Z(\mathcal{A})$. The element z of $Z(\mathcal{A})$ corresponding to a given basis element \bar{b} is such that $\varepsilon(bz) = 1$ for any $b \in \mathcal{A}$ with $\pi(b) = \bar{b}$, $\pi : \mathcal{A} \rightarrow \mathcal{A}/[\mathcal{A}, \mathcal{A}]_t$ being the projection map. To find these elements of $Z(\mathcal{A})$, we first need to find the elements of the coset of \mathcal{A} corresponding to a given element of $\mathcal{A}/[\mathcal{A}, \mathcal{A}]_t$. The element of $Z(\mathcal{A})$ we want is then the sum of the complementary elements of the elements of that coset. Here is an example for $n = 3$. Take the identity element in $\mathcal{A}/[\mathcal{A}, \mathcal{A}]_t$ represented by three non-intersecting strands on the Möbius band. Its coset in \mathcal{A} is just itself: T_e . Its complementary element is T_{\max_n} , our first basis element for $Z(\mathcal{A})$. Conversely, the coset of $T_{\max_n} \in \mathcal{A}/[\mathcal{A}, \mathcal{A}]_t$ is again just $T_{\max_n} \in \mathcal{A}$ and its complementary element is the identity $T_e \in Z(\mathcal{A})$, our second basis element. Now let us consider $T_1 \in \mathcal{A}/[\mathcal{A}, \mathcal{A}]_t$. Its coset is $\{T_1, T_2\} \subset \mathcal{A}$ since T_1 is equivalent to T_2 in $\mathcal{A}/[\mathcal{A}, \mathcal{A}]_t$. Their complementary elements are $T_{s_2 s_1} = T_2 T_1$ and

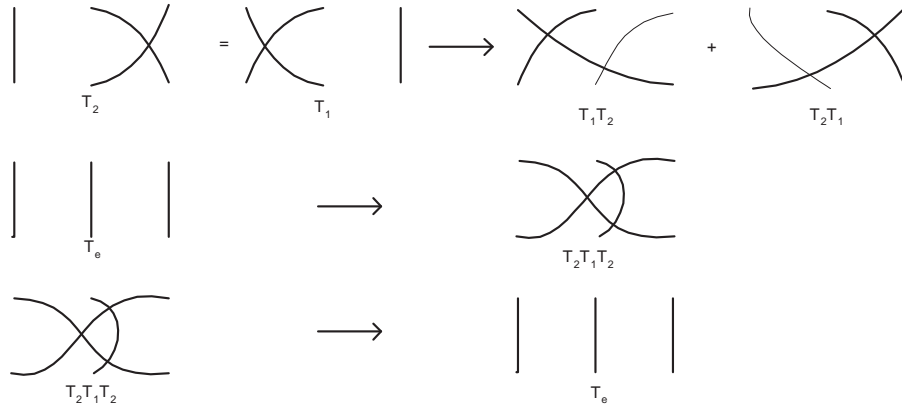


Figure 18: This figures shows the cosets corresponding to diagrams on the Möbius band of total thickness three and the basis elements of $Z(\mathcal{NC}_n)$ formed by taking the sum of their complements.

$T_{s_1 s_2} = T_1 T_2$ respectively. Our third basis element is therefore $T_2 T_1 + T_1 T_2$. This ends our computation of the basis of $Z(\mathcal{A})$ for $n = 3$. See Figure 18 for the diagrams corresponding to this computation.

The Nilcoxeter algebra is a graded algebra with the grading given by the length function on \mathcal{S}_n . Equivalently, the grading of a basis element is given by the number of crossings in its corresponding diagram. Note that the process described above to find a basis for $Z(\mathcal{A})$ yields a homogeneous basis since every element of the coset of an element of $\mathcal{A}/[\mathcal{A}, \mathcal{A}]_t$ has the same length, or the same number of crossings.

4.3 Multiplication in the Center of \mathcal{NC}_n

We would now like to know what the multiplication table looks like in the basis we have described in the previous section.

Lemma 4.1. *Every diagram corresponding to a basis element of $\mathcal{A}/[\mathcal{A}, \mathcal{A}]_t$ other than T_{\max_n} has a vertical line as its first or last strand. Equivalently, a basis element of $\mathcal{A}/[\mathcal{A}, \mathcal{A}]_t$ other than T_{\max_n} cannot have both T_1 and T_{n-1} in its expression.*

Proof. First, let us look at a non-zero diagram D on the Möbius band corresponding to a partition of n which has at least one even part. Then, we know that one of these even prime diagrams is on the outside of D and disjoint from its other components. We also know that all the elements in the coset of this prime even diagram have at least one such vertical strand as either their first or last strand. Since this even component is disjoint from all the other components of the diagram, this vertical strand clearly remains in all elements of the coset of D . Figure 19 shows this for all the diagrams in the coset corresponding to the partition $6 = 4 + 1 + 1$.

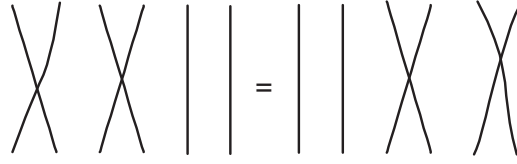


Figure 19: These are the diagrams in the coset corresponding to the partition $6 = 4 + 1 + 1$. Note that the even component of thickness four is always disjoint from the odd components of thickness one. Since the prime diagram of thickness four always has a vertical strand either as its first or last strand, so do all these composite diagrams.

We now look at a non-zero diagram D on the Möbius band which is composed solely of odd prime components. Since D is not the diagram for T_{max_n} , one of its prime components has thickness greater than one. Let the thickest prime component C of D have thickness $2k + 1$. Remember from the classification of these diagram that this means that this component C must be the component occupying positions 1 and n both at the top and at the bottom of the diagram. This is true for any diagram equivalent to D . Once again, every diagram of an odd prime component must have a vertical strand at position 1 or $2k + 1$. This is true also of C in D since C occupies positions 1 and n both at the top and bottom of the diagram. Figure 20 shows that this is true in the case of the diagrams corresponding to the partition $4 = 3 + 1$. \square

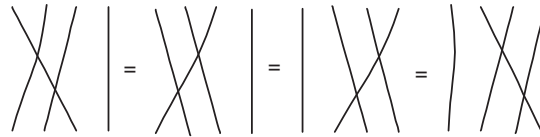


Figure 20: These are the diagrams in the coset corresponding to the partition $4 = 3 + 1$. Note that the prime component of thickness three is on the outside of the odd component of thickness one. Since the prime diagram of thickness three always has a vertical strand either as its first or last strand, so do all these composite diagrams.

Proposition 4.2. *The multiplication table for the center of \mathcal{NC}_n in the basis described above is trivial.*

Proof. This vertical strand in every element of the cosets of our basis elements means that their complement must have either a strand going from top position 1 to bottom position n or from top position n to bottom position 1, depending on whether the vertical strand is at position 1 or n , since T_{max_n} has both of these diagonals. Since every basis element of $Z(\mathcal{A})$ is a sum of such complements, every diagram appearing in any basis element of $Z(\mathcal{A})$ has such a diagonal.

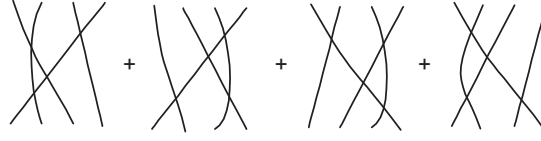


Figure 21: This is the basis element of $Z(\mathcal{NC}_n)$ for $n = 4$ corresponding to the partition $4 = 3 + 1$. It is the sum of the complements of the elements in the corresponding coset in \mathcal{NC}_n .

Multiplying two elements with the opposite diagonals obviously yields a non-minimal, and therefore zero, diagram since the strand in the product corresponding to these diagonals crosses every other strand twice.

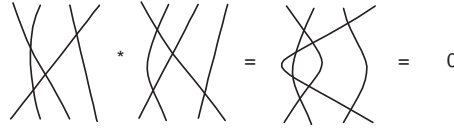


Figure 22: This is an example of the multiplication of two elements of \mathcal{NC}_n with opposite full diagonals. Such a product is always zero.

If we multiply two elements with the same diagonal, we also have a trivial diagram. Indeed, remember that if two strands cross more than once, the diagram is zero. Since this diagonal strand crosses all other strands in the first element and does not remain vertical in the second because of the full diagonal of this second diagram, it must cross at least one strand twice.

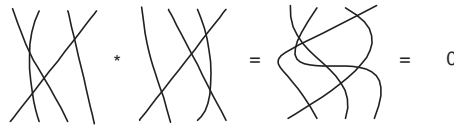


Figure 23: This is an example of the multiplication of two elements of \mathcal{NC}_n with the same full diagonal. Such a product is always zero since the strand corresponding to the diagonal in the first element crosses at least one strand twice.

Hence, the product of any basis element by any element other than the identity is zero and the multiplication table of $Z(\mathcal{A})$ is trivial as claimed.

□

5 The Center of the 0-Hecke Algebra

5.1 0-Hecke Diagrams on the Möbius band

The classification of \mathcal{H}_n diagrams on the Möbius band is identical to that of \mathcal{NC}_n diagrams. All of the arguments of Section 3 go through for \mathcal{H}_n . Everytime a diagram was eliminated for being zero in \mathcal{NC}_n , it is eliminated for not being minimal in \mathcal{H}_n . Zero diagrams in \mathcal{NC}_n correspond exactly to non-minimal diagrams in \mathcal{H}_n . Non-zero diagrams which are equivalent in \mathcal{NC}_n are also equivalent in \mathcal{H}_n and non-minimal diagrams which are equivalent in \mathcal{H}_n are equivalent in \mathcal{NC}_n . Hence, the vector spaces $\mathcal{A}/[\mathcal{A}, \mathcal{A}]_t$ for $\mathcal{A} = \mathcal{NC}_n$ and $\mathcal{A} = \mathcal{H}_n$ are isomorphic and their bases are represented by the same diagrams on the Möbius band.

Therefore, the dimensions of the centers of \mathcal{NC}_n and \mathcal{H}_n are the same:

$$\dim(Z) = \sum_{\lambda \vdash n} \frac{n_\lambda!}{m_\lambda},$$

where n_λ is the number of even parts in λ and $m_\lambda = \prod_j (i_j!)$.

5.2 A Basis for the Center

Finding the basis for the center of \mathcal{H}_n corresponding to our basis for $\mathcal{A}/[\mathcal{A}, \mathcal{A}]_t$ is more complex than in the case of the Nilcoxeter algebra. In that previous case, it was easy to see what linear combination of diagrams would complement any diagram representing a given basis element of $\mathcal{A}/[\mathcal{A}, \mathcal{A}]_t$. In other words, given a basis element $\bar{b} \in \mathcal{A}/[\mathcal{A}, \mathcal{A}]_t$, we needed to find an element $z \in Z(\mathcal{A})$ such that $\varepsilon(bz) = 1$ for any $b \in \mathcal{A}$ with $\pi(b) = \bar{b} \in \mathcal{A}/[\mathcal{A}, \mathcal{A}]_t$. This z was simply the sum of the complements of all such $b \in \mathcal{A}$.

Because of the relation

$$T_i^2 = T_i,$$

the coset of an element of $\mathcal{A}/[\mathcal{A}, \mathcal{A}]_t$ tends to be much bigger for $\mathcal{A} = \mathcal{H}_n$ than for $\mathcal{A} = \mathcal{NC}_n$. To generate all of the elements of this coset, one needs to first generate all diagrams which are equivalent on the Möbius band, just as was done for \mathcal{NC}_n . Then, use the above relation to double crossings and again generate all equivalent diagrams. Repeat this process until all the new diagrams are redundant. This algorithm generates redundant diagrams which should be ignored.

Then, an element of \mathcal{H}_n may have more than one complement. However, note that a diagram in \mathcal{H}_n can only have one complement with a given number of crossings. It then becomes clear that a simple sum of complements like the one we had for \mathcal{NC}_n will not do; one needs a more complicated linear combination of them. Figure 24 gives the answer for $n = 3$.

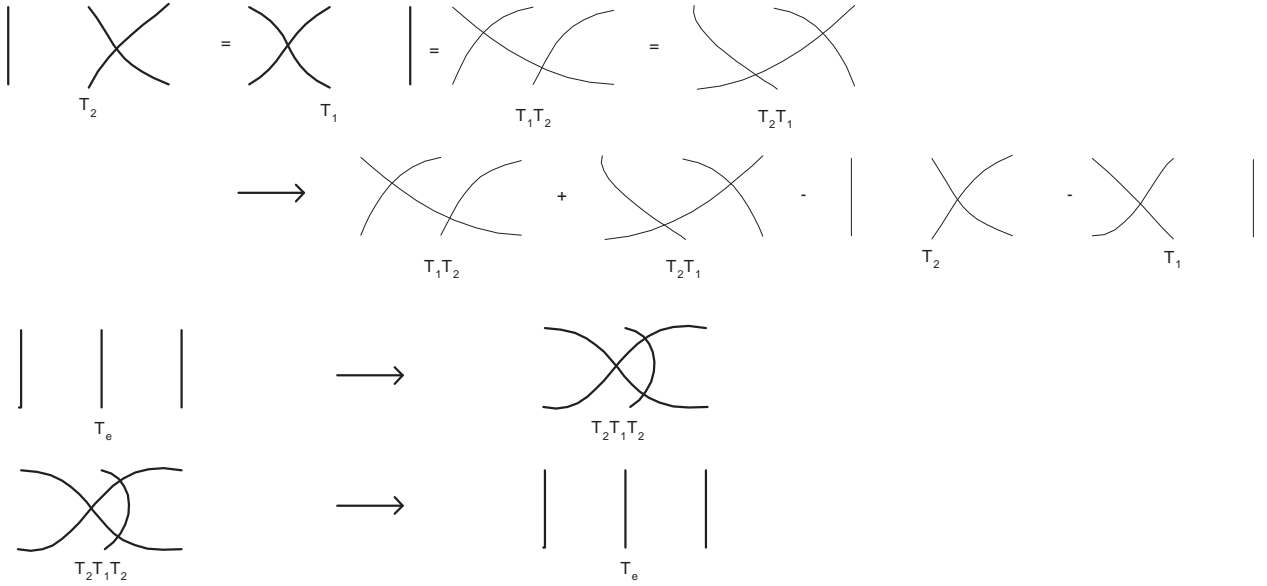


Figure 24: This figures shows the cosets corresponding to diagrams on the Möbius band of total thickness three and the basis elements of $Z(\mathcal{H}_n)$ formed by taking the sum of their complements.

References

- [1] J.N. Bernstein, I.M. Gelfand, and S.I. Gelfand. Schubert cells and cohomology of the spaces G/P . *Russian Math. Surveys*, 28:1–26, 1973.
- [2] R.W. Carter. Representation theory of the 0-Hecke algebra. *J. Algebra*, 104:89–103, 1986.
- [3] G. Duchamp, F. Hivert, and J.-Y. Thibon. Noncommutative symmetric functions VI: Free quasi-symmetric functions and related algebras, 2001.
- [4] S. Fomin and R.P. Stanley. Schubert polynomials and the nilCoxeter algebra. *Advances in Math.*, 103:196–207, 1994.
- [5] James E. Humphrey. *Reflection Groups and Coxeter Groups*, volume 29 of *Cambridge studies in advanced mathematics*. Cambridge University Press, Cambridge, UK, 1990.
- [6] D. Kazhdan and G. Lusztig. Representations of Coxeter groups and Hecke algebras. *Invent. Math.*, 53:165–184, 1979.
- [7] M. Khovanov. Nilcoxeter algebras categorify the Weyl algebra. *COMM.ALGEBRA* 29, 11:5033, 2001.

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- [8] M. Khovanov and A. D. Lauda. A diagrammatic approach to categorification of quantum groups i, 2008.
 - [9] D. Krob and J.-Y. Thibon. Noncommutative symmetric functions IV. quantum linear groups and hecke algebras at $q = 0$. *J. Algebraic Combin.*, 6:339–376, 1997.
 - [10] A. Lascoux and M.-P. Schützenberger. Fonctorialité des polynômes de Schubert In *Proceedings of the AMS Special Session on Invariant Theory, Vol. 88 (Denton, Texas, 1986)*, pages 585–595, Providence, RI, 1987. Amer. Math. Soc.
 - [11] I.G. Macdonald. *Notes on Schubert polynomials*. Université du Québec à Montréal, Montréal, QC, Canada, 1991.
 - [12] P.N. Norton. 0-Hecke algebras. *J. Austral. Math. Soc.*, A27:337–357, 1979.